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# HARMONIC MOMENTS AND LARGE DEVIATIONS FOR A SUPERCRITICAL BRANCHING PROCESS IN A RANDOM ENVIRONMENT

ION GRAMA, QUANSHENG LIU, AND ERIC MIQUEU

ABSTRACT. Let  $(Z_n)$  be a supercritical branching process in an independent and identically distributed random environment  $\xi$ . We study the asymptotic of the harmonic moments  $\mathbb{E}[Z_n^{-r}|Z_0 = k]$  of order  $r > 0$  as  $n \rightarrow \infty$ . We exhibit a phase transition with the critical value  $r_k > 0$  determined by the equation  $\mathbb{E}p_1^k = \mathbb{E}m_0^{-r_k}$ , where  $m_0 = \sum_{k=0}^{\infty} kp_k$  with  $p_k = \mathbb{P}(Z_1 = k|\xi)$ , assuming that  $p_0 = 0$ . Contrary to the constant environment case (the Galton-Watson case), this critical value is different from that for the existence of the harmonic moments of  $W = \lim_{n \rightarrow \infty} Z_n/\mathbb{E}(Z_n|\xi)$ . The aforementioned phase transition is linked to that for the rate function of the lower large deviation for  $Z_n$ . As an application, we obtain a lower large deviation result for  $Z_n$  under weaker conditions than in previous works and give a new expression of the rate function. We also improve an earlier result about the convergence rate in the central limit theorem for  $W - W_n$ , and find an equivalence for the large deviation probabilities of the ratio  $Z_{n+1}/Z_n$ .

## 1. INTRODUCTION

A branching process in a random environment (BPRE) is a natural and important generalisation of the Galton-Watson process, where the reproduction law varies according to a random environment indexed by time. It was introduced for the first time in Smith and Wilkinson [24] to modelize the growth of a population submitted to an environment. For background concepts and basic results concerning a BPRE we refer to Athreya and Karlin [4, 3]. In the critical and subcritical regime the branching process goes out and the research interest has been mostly concentrated on the survival probability and conditional limit theorems, see e.g. Afanasyev, Böinghoff, Kersting, Vatutin [1, 2], Vatutin [26], Vatutin and Zheng [27], and the references therein. In the supercritical case, a great deal of current research has been focused on large deviations, see e.g. Bansaye and Berestycki [6], Bansaye and Böinghoff [7, 8, 9], Böinghoff and Kersting [11], Huang and Liu [17] and Nakashima [22]. In the particular case when the offspring distribution has a fractional linear generating function, precise asymptotics can be found in Böinghoff [10] and Kozlov [19]. An important closely linked issue is the asymptotic behavior of the harmonic

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moments  $\mathbb{E}[Z_n^{-r}|Z_0 = k]$  of the process  $Z_n$  starting with  $Z_0 = k$  initial individuals. For the Galton-Watson process which corresponds to the constant environment case, the question has been studied exhaustively in Ney and Vidyashankar [23]. For a BPPE, it has only been partially treated in [17, Theorem 1.3].

In the present paper, we give a complete description of the asymptotic behavior of the harmonic moments  $\mathbb{E}_k[Z_n^{-r}] = \mathbb{E}[Z_n^{-r}|Z_0 = k]$  of the process  $Z_n$  starting with  $k$  individuals and assuming that each individual gives birth to at least one offspring (non-extinction case). As a consequence, we improve the lower large deviation result for the process  $Z_n$  obtained in [8, Theorem 3.1] by relaxing the hypothesis therein. In the meanwhile we give a new characterization of the rate function in the large deviation result stated in [8]. We also improve the exponential convergence rate in the central limit theorem for  $W - W_n$  established in [17]. Furthermore, we investigate the large deviation behavior of the ratio  $R_n = \frac{Z_{n+1}}{Z_n}$ , i.e. the asymptotic of the large deviation probability  $\mathbb{P}(|R_n - m_n| > a)$  for  $a > 0$ , where  $m_n$  is the expected value of the number of children of an individual in generation  $n$  given the environment  $\xi$ . For the Galton-Watson process, the quantity  $R_n$  is the Lotka-Nagaev estimator of the mean  $\mathbb{E}Z_1$ , whose large deviation probability has been studied in [23].

Let us explain briefly the findings of the paper in the special case when we start with  $Z_0 = 1$  individual. Assume that  $\mathbb{P}(Z_1 = 0) = 0$  and  $\mathbb{P}(Z_1 = 1) > 0$ . Define  $r_1$  as the solution of the equation

$$(1.1) \quad \mathbb{E}m_0^{-r_1} = \gamma,$$

with

$$(1.2) \quad \gamma = \mathbb{P}(Z_1 = 1).$$

From Theorem 2.1 we get the following asymptotic behavior of the harmonic moments  $\mathbb{E}[Z_n^{-r}]$  for  $r > 0$ . Assume that  $\mathbb{E}m_0^{r_1+\varepsilon} < \infty$  for some  $\varepsilon > 0$ . Then, we have

$$(1.3) \quad \begin{cases} \frac{\mathbb{E}[Z_n^{-r}]}{\gamma^n} \xrightarrow{n \rightarrow \infty} C(r) & \text{if } r > r_1, \\ \frac{\mathbb{E}[Z_n^{-r}]}{n\gamma^n} \xrightarrow{n \rightarrow \infty} C(r) & \text{if } r = r_1, \\ \frac{\mathbb{E}[Z_n^{-r}]}{(\mathbb{E}m_0^{-r})^n} \xrightarrow{n \rightarrow \infty} C(r) & \text{if } r < r_1, \end{cases}$$

where  $C(r)$  are positive constants for which we find integral expressions. This shows that there are three phase transitions in the rate of convergence of the harmonic moments for the process  $Z_n$ , with the critical value  $r_1$ . It generalizes the result of [23] for the Galton-Watson process. For a BPPE, it completes and improves the result of [17], where the asymptotic equivalent of the quantity  $\mathbb{E}[Z_n^{-r}]$  has been established in the particular case where  $r < r_1$  and under stronger assumptions.

The proof presented here is new and straightforward compared to that for the Galton-Watson process given in [23]. Indeed, we prove (1.3) starting from the branching property

$$(1.4) \quad Z_{n+m} = \sum_{i=1}^{Z_m} Z_{n,i}^{(m)},$$

where conditionally on the environment  $\xi$ , for  $i \geq 1$ , the sequences of random variables  $\{Z_{n,i}^{(m)} : n \geq 0\}$  are i.i.d. branching processes with the shifted environment  $T^m(\xi_0, \xi_1, \dots) = (\xi_m, \xi_{m+1}, \dots)$ , and are also independent of  $Z_m$ . This simple idea leads to the following equation which will play a key role in our arguments:

$$(1.5) \quad \mathbb{E}[Z_{n+1}^{-r}] = \gamma^{n+1} + \sum_{j=0}^n b_j \gamma^{n-j} c_r^j,$$

where  $c_r = \mathbb{E}m_0^{-r}$  and  $(b_j)_{j \geq 0}$  is an increasing and bounded sequence. Such a relation highlights the main role played by the quantities  $\gamma$  and  $c_r$  in the asymptotic study of  $\mathbb{E}[Z_n^{-r}]$  whose behavior depends on whether  $\gamma < c_r$ ,  $\gamma = c_r$  or  $\gamma > c_r$ . Note that the complete proof of (1.3) relies on some recent and important results established in [14] concerning the critical value for the existence of the harmonic moments of the r.v.  $W$  and the asymptotic behavior of the distribution  $\mathbb{P}(Z_n = j)$  as  $n \rightarrow \infty$ , with  $j \geq 1$ . For the Galton-Watson process, our approach based on (1.5) is much simpler than that in [23].

Our proof also gives an expression of the limit constants in (1.3). For the Galton-Watson process, it recovers the expressions of [23, Theorem 1] in the cases where  $\gamma > c_r$  and  $\gamma < c_r$ . In the critical case where  $r = r_1$ , the limit constant obtained in this paper is different to that of [23, Theorem 1], which leads to an alternative expression of the constant and the following nice identity involving the well-known functions  $G$ ,  $Q$  and  $\phi$ : defining

$$\begin{aligned} G(t) &= \sum_{k=0}^{\infty} t^k \mathbb{P}(Z_1 = k), \\ Q(t) &= \lim_{n \rightarrow \infty} \gamma^{-n} G^{\circ n}(t), \\ \phi(t) &= \mathbb{E}[e^{-tW}], \end{aligned}$$

and denoting  $m = \mathbb{E}[Z_1]$ ,  $\gamma = \mathbb{P}(Z_1 = 1)$  and  $\bar{G}(t) = G(t) - \gamma t$ , we have

$$(1.6) \quad \frac{1}{\gamma} \int_0^{\infty} \bar{G}(\phi(u)) u^{r-1} du = \int_1^m Q(\phi(u)) u^{r-1} du.$$

For a BPRE, we will show a generalization of (1.6) in Proposition 2.2.

As a consequence of Theorem 2.1 and of a version of the Gärtner-Ellis theorem, we obtain a lower large deviation result for  $Z_n$  under conditions weaker than those in [8, Theorem 3.1]. Assume that  $\mathbb{P}(Z_1 = 0) = 0$  and  $\mathbb{E}m_0^{r_1+\varepsilon} < \infty$  for some  $\varepsilon > 0$ . Let

$$(1.7) \quad \Lambda(\lambda) = \log \mathbb{E}e^{\lambda X}$$

be the log-Laplace transform of  $X = \log m_0$  and

$$(1.8) \quad \Lambda^*(x) = \sup_{\lambda \leq 0} \{\lambda x - \Lambda(\lambda)\}$$

be the Fenchel-Legendre transform of  $\Lambda(\cdot)$ . Then, for any  $\theta \in (0, \mathbb{E}[X])$ , we have

$$(1.9) \quad \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(Z_n \leq e^{\theta n}) = \chi^*(\theta) \in (0, \infty),$$

where

$$(1.10) \quad \chi^*(\theta) = \begin{cases} -r_1\theta - \log \gamma & \text{if } 0 < \theta < \theta_1, \\ \Lambda^*(\theta) & \text{if } \theta_1 \leq \theta < \mathbb{E}[X], \end{cases}$$

with

$$(1.11) \quad \theta_1 = \Lambda'(-r_1) \in (0, \mathbb{E}[X]).$$

Equation (1.9) improves the result of [8, Theorem 3.1(ii)] in the case when  $\mathbb{P}(Z_1 = 0) = 0$ , since it is assumed in [8] that  $\mathbb{E}m_0^t < \infty$  for all  $t > 0$ , whereas we only require that  $\mathbb{E}m_0^{r_1+\varepsilon} < \infty$  for some  $\varepsilon > 0$ . Moreover, equations (1.10) and (1.11) also give new and alternative expressions of the rate function and the critical value. In fact, it has been proved in [8] that, in the case when  $\mathbb{P}(Z_1 = 0) = 0$  and  $Z_0 = 1$ ,

$$(1.12) \quad \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(Z_n \leq e^{\theta n}) = I(\theta) \in (0, \infty),$$

with

$$(1.13) \quad I(\theta) = \begin{cases} \rho \left(1 - \frac{\theta}{\theta_1^*}\right) + \frac{\theta}{\theta_1^*} \Lambda^*(\theta_1^*) & \text{if } 0 < \theta < \theta_1^*, \\ \Lambda^*(\theta) & \text{if } \theta_1^* \leq \theta < \mathbb{E}[X], \end{cases}$$

where  $\rho = -\log \gamma$  and  $\theta_1^*$  the unique solution on  $(0, \mathbb{E}[X])$  of the equation

$$(1.14) \quad \frac{\rho - \Lambda^*(\theta_1^*)}{\theta_1^*} = \inf_{0 \leq \theta \leq \mathbb{E}[X]} \frac{\rho - \Lambda^*(\theta)}{\theta}.$$

It follows directly from the relations (1.10) to (1.14) that  $\theta_1 = \theta_1^*$  and  $\chi^*(\theta) = I(\theta)$  for all  $\theta \in (0, \mathbb{E}[X])$ . This fact can also be shown by using simple duality arguments between  $\Lambda$  and  $\Lambda^*$ , as will be seen in the next section.

The rest of the paper is organized as follows. In Section 2 we give the precise statements of the main theorems with applications. Section 3 is devoted to the proof of the main results, Theorems 2.1 and 2.3. The proofs for the applications are deferred to Section 4.

Throughout the paper, we denote by  $C$  an absolute constant whose value may differ from line to line.

## 2. MAIN RESULTS

A BPPE ( $Z_n$ ) can be described as follows. The random environment is represented by a sequence  $\xi = (\xi_0, \xi_1, \dots)$  of independent and identically distributed random variables (i.i.d. r.v.'s) taking values in an abstract space  $\Xi$ , whose realizations determine the probability generating functions

$$(2.1) \quad f_{\xi_n}(s) = f_n(s) = \sum_{i=0}^{\infty} p_i(\xi_n) s^i, \quad s \in [0, 1], \quad p_i(\xi_n) \geq 0, \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1.$$

The BPPE  $(Z_n)_{n \geq 0}$  is defined by the relations

$$(2.2) \quad Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} N_{n,i}, \quad \text{for } n \geq 0,$$

where the random variables  $N_{n,i}$  ( $i = 1, 2, \dots$ ) represent the number of children of the  $i$ -th individual of the generation  $n$ . Conditionally on the environment  $\xi$ , the r.v.'s  $N_{n,i}$  ( $n \geq 0, i \geq 1$ ) are independent of each other, and each  $N_{n,i}$  ( $i \geq 1$ ) has common probability generating function  $f_n$ .

In the sequel we denote by  $\mathbb{P}_\xi$  the *quenched law*, i.e. the conditional probability when the environment  $\xi$  is given, and by  $\tau$  the law of the environment  $\xi$ . Then  $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx) \tau(d\xi)$  is the total law of the process, called *annealed law*. The corresponding quenched and annealed expectations are denoted respectively by  $\mathbb{E}_\xi$  and  $\mathbb{E}$ . We also denote by  $\mathbb{P}_k$  and  $\mathbb{E}_k$  the corresponding annealed probability and expectation starting with  $Z_0 = k$  individuals, with  $\mathbb{P}_1 = \mathbb{P}$  and  $\mathbb{E}_1 = \mathbb{E}$ . From (2.2), it follows that the probability generating function of  $Z_n$  conditionally on the environment  $\xi$  is given by

$$(2.3) \quad g_n(t) = \mathbb{E}_\xi[t^{Z_n}] = f_0 \circ \dots \circ f_{n-1}(t), \quad 0 \leq t \leq 1.$$

Since  $\xi_0, \xi_1, \dots$  are i.i.d. r.v.'s, we get that the annealed probability generating function  $G_{k,n}$  of  $Z_n$  starting with  $Z_0 = k$  individuals is given by

$$(2.4) \quad G_{k,n}(t) = \mathbb{E}_k[t^{Z_n}] = \mathbb{E}[g_n^k(t)], \quad 0 \leq t \leq 1.$$

We also define, for  $n \geq 0$ ,

$$m_n = m(\xi_n) = \sum_{i=0}^{\infty} i p_i(\xi_n) \quad \text{and} \quad \Pi_n = \mathbb{E}_\xi Z_n = m_0 \dots m_{n-1},$$

where  $m_n$  represents the average number of children of an individual of generation  $n$  when the environment  $\xi$  is given, and  $\Pi_0 = 1$  by convention. Let

$$(2.5) \quad W_n = \frac{Z_n}{\Pi_n}, \quad n \geq 0,$$

be the normalized population size. It is well known that under the quenched law  $\mathbb{P}_\xi$ , as well as under the annealed law  $\mathbb{P}$ , the sequence  $(W_n)_{n \geq 0}$  is a non-negative martingale with respect to the filtration

$$\mathcal{F}_n = \sigma(\xi, N_{k,i}, 0 \leq k \leq n-1, i = 1, 2, \dots),$$

where by convention  $\mathcal{F}_0 = \sigma(\xi)$ . Then the limit  $W = \lim W_n$  exists  $\mathbb{P}$  - a.s. and  $\mathbb{E}W \leq 1$ . We also denote the quenched and annealed Laplace transform of  $W$  by

$$(2.6) \quad \phi_\xi(t) = \mathbb{E}_\xi \left[ e^{-tW} \right] \quad \text{and} \quad \phi(t) = \mathbb{E} \left[ e^{-tW} \right], \quad \text{for } t \geq 0,$$

while starting with  $Z_0 = 1$  individual. For  $k \geq 1$ , while starting with  $Z_0 = k$  individuals, we have

$$(2.7) \quad \phi_k(t) := \mathbb{E}_k \left[ e^{-tW} \right] = \mathbb{E}[\phi_\xi^k(t)].$$

Another important tool in the study of a BPRE is the associated random walk

$$S_n = \log \Pi_n = \sum_{i=1}^n X_i, \quad n \geq 1,$$

where the r.v.'s  $X_i = \log m_{i-1}$  ( $i \geq 1$ ) are i.i.d. depending only on the environment  $\xi$ . For the sake of brevity, we set  $X = \log m_0$  and

$$\mu = \mathbb{E}X.$$

We shall consider a supercritical BPRE where  $\mu \in (0, \infty)$ , so that under the extra condition  $\mathbb{E}|\log(1 - p_0(\xi_0))| < \infty$  (see [24]), the population size tends to infinity with positive probability. For our propose, in fact we will assume in the whole paper that each individual gives birth to at least one child, i.e.

$$(2.8) \quad p_0(\xi_0) = 0 \quad \text{a.s.}$$

Therefore, under the condition

$$(2.9) \quad \mathbb{E} \frac{Z_1}{m_0} \log^+ Z_1 < \infty,$$

the martingale  $(W_n)$  converges to  $W$  in  $L^1(\mathbb{P})$  (see e.g. [25]) and

$$\mathbb{P}(W > 0) = \mathbb{P}(Z_n \rightarrow \infty) = 1.$$

Now we can state the main result of the paper about the asymptotic of the harmonic moments  $\mathbb{E}_k[Z_n^{-r}]$  of the process  $(Z_n)$  for  $r > 0$ , starting with  $Z_0 = k$  for  $k \geq 1$ . Define

$$(2.10) \quad \gamma_k = \mathbb{P}_k(Z_1 = k) = \mathbb{E}[p_1^k(\xi_0)]$$

and

$$(2.11) \quad \bar{G}_{k,1}(t) = G_{k,1}(t) - \gamma_k t^k = \sum_{j=k+1}^{\infty} t^j \mathbb{P}(Z_1 = j),$$

where  $G_{k,1}$  is the generating function of  $Z_1$  defined in (2.4). Let  $r_k$  be the solution of the equation

$$(2.12) \quad \gamma_k = \mathbb{E} \left[ m_0^{-r_k} \right],$$

with the convention that  $r_k = +\infty$  if  $p_1(\xi_0) = 0$  a.s. For any  $r > 0$ , set

$$(2.13) \quad c_r = \mathbb{E} m_0^{-r}.$$

For any  $k \geq 1$  and  $r > 0$ , let  $\mathbb{P}_k^{(r)}$  be the probability measure (depending on  $r$ ) defined, for any  $\mathcal{F}_n$ -measurable r.v.  $T$ , by

$$(2.14) \quad \mathbb{E}_k^{(r)}[T] = \frac{\mathbb{E}_k[\Pi_n^{-r}T]}{c_r^n}.$$

Set  $\mathbb{P}^{(r)} = \mathbb{P}_1^{(r)}$  and  $\mathbb{E}^{(r)} = \mathbb{E}_1^{(r)}$ . It is easily seen that under  $\mathbb{P}^{(r)}$ , the process  $(Z_n)$  is still a supercritical branching process in a random environment with  $\mathbb{P}^{(r)}(Z_1 = 0) = 0$ , and  $(W_n)$  is still a non-negative martingale which converges a.s. to  $W$ . Moreover, by (2.9) and the fact that  $m_0 \geq 1$ , we have

$$\mathbb{E}^{(r)}\left[\frac{Z_1}{m_0} \log Z_1\right] = \mathbb{E}\left[\frac{Z_1}{m_0^{1+r}} \log Z_1\right] / \mathbb{E}[m_0^{-r}] \leq \mathbb{E}\left[\frac{Z_1}{m_0} \log Z_1\right] / \mathbb{E}[m_0^{-r}] < \infty,$$

which implies that

$$(2.15) \quad W_n \rightarrow W \quad \text{in} \quad L^1(\mathbb{P}^{(r)}).$$

Then we have the following equivalent for the harmonic moments of  $Z_n$ . Let

$$(2.16) \quad A_{k,n}(r) = \begin{cases} \gamma_k^n, & \text{if } r > r_k, \\ n\gamma_k^n, & \text{if } r = r_k, \\ c_r^n, & \text{if } r < r_k. \end{cases}$$

**Theorem 2.1.** *Let  $k \geq 1$  and assume that  $\mathbb{E}m_0^{r_k+\varepsilon} < +\infty$  for some  $\varepsilon > 0$ . Then we have*

$$(2.17) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}_k[Z_n^{-r}]}{A_{k,n}(r)} = C(k, r) := \begin{cases} \frac{1}{\Gamma(r)} \int_0^\infty Q_k(e^{-t}) t^{r-1} dt & \text{if } r > r_k, \\ \frac{\gamma_k^{-1}}{\Gamma(r)} \mathbb{E}^{(r)} \left[ \int_0^\infty \bar{G}_{k,1}(\phi_\xi(t)) t^{r-1} dt \right] & \text{if } r = r_k, \\ \frac{1}{\Gamma(r)} \int_0^\infty \phi_k^{(r)}(t) t^{r-1} dt & \text{if } r < r_k, \end{cases}$$

where  $C(k, r) \in (0, \infty)$ ,  $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$  is the Gamma function, and  $\phi_k^{(r)}(t) = \mathbb{E}_k^{(r)}[e^{-tW}]$  is the Laplace transform of  $W$  under  $\mathbb{P}_k^{(r)}$ .

This theorem shows that there is a phase transition in the rate of convergence of the harmonic moments of  $Z_n$  with the critical value  $r_k > 0$  defined by (2.12). This critical value  $r_k$  is generally different from the critical value  $a_k$  for the existence of the harmonic moment of  $W$ . Indeed, as shown in [14, Theorem 2.1] (see Lemma 3.1 below), the critical value  $a_k$  is determined by

$$(2.18) \quad \mathbb{E}[p_1^k(\xi_0) m_0^{a_k}] = 1,$$

which is in general different from the critical value  $r_k$  determined by (2.12), that is

$$(2.19) \quad \mathbb{E}[p_1^k(\xi_0)] = \mathbb{E}[m_0^{-r_k}].$$



This is in contrast with the Galton-Watson process where  $r_k = a_k = r_1^k$ , with  $r_1$  the solution of the equation  $p_1 m_0^{r_1} = 1$  which coincides with both equations (2.18) and (2.19). Theorem 2.1 generalizes the result of [23] for the Galton-Watson process. For a BPPE, it completes and improves Theorem 1.3 of [17], where the formula (2.17) was only proved for  $k = 1$  and  $r < r_1$ , and under the following much stronger boundedness condition: there exist some constants  $p, c_0, c_1 > 1$  such that

$$c_0 \leq m_0 \leq c_1 \quad \text{and} \quad c_0 \leq m_0(p) \leq c_1 \quad a.s.,$$

where  $m_0(p) = \sum_{i=1}^{\infty} i^p p_i(\xi_0)$ . Instead of this boundedness condition, here we only require the moment assumption  $\mathbb{E}[m_0^{r_k+\varepsilon}] < \infty$  for some  $\varepsilon > 0$ .

For the Galton-Watson process with  $k = 1$  initial individuals, the expression of the limit constant in the case when  $r = r_1$  (up to the constant factor  $\Gamma(r)$ ) becomes

$$\frac{1}{\gamma} \int_0^\infty \bar{G}(\phi(u)) u^{r-1} du,$$

whereas it has been proved in [23] that the limit constant is equal to

$$\int_1^m Q(\phi(u)) u^{r-1} du.$$

Actually, the above two expressions coincide, as shown by the following result valid for a general BPPE. Let  $Q_k(t)$  be defined by

$$(2.20) \quad Q_k(t) = \lim_{n \rightarrow \infty} \frac{G_{k,n}(t)}{\gamma_k^n} = \sum_{j=k}^{\infty} q_{k,j} t^j, \quad t \in [0, 1),$$

where  $G_{k,n}$  is defined by (2.4) and the limit exists according to [14, Theorem 2.3] (see Lemma 3.2 below).

**Proposition 2.2.** *For  $k \geq 1$  and  $r = r_k$ , we have*

$$(2.21) \quad \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \int_0^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right] = \mathbb{E}^{(r)} \left[ \int_1^{m_0} Q_k(\phi_\xi(u)) u^{r-1} du \right].$$

As an application of Theorem 2.1 we get a large deviation result. Indeed,  $\mathbb{E}[Z_n^\lambda] = \mathbb{E}[e^{\lambda \log Z_n}]$  is the Laplace transform of  $\log Z_n$ . From Theorem 2.1 we obtain

$$(2.22) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_k[Z_n^\lambda] = \chi_k(\lambda) = \begin{cases} \log \gamma_k & \text{if } \lambda \leq \lambda_k, \\ \Lambda(\lambda) & \text{if } \lambda \in [\lambda_k, 0]. \end{cases}$$

Thus using a version of the Gärtner-Ellis theorem adapted to the study of tail probabilities (see [17, Lemma 3.1]), we get the following lower large deviation result for the BPPE  $(Z_n)$ . Recall that  $\Lambda(\lambda) = \log \mathbb{E} e^{\lambda X}$  is the log-Laplace of  $X = \log m_0$ , and  $\Lambda^*(\cdot)$  is the Fenchel-Legendre transform of  $\Lambda(\cdot)$  defined in (1.8).

**Theorem 2.3.** *Let  $k \geq 1$  and  $r_k$  be the solution of the equation (2.12). Assume that  $\mathbb{E} m_0^{r_k+\varepsilon} < \infty$  for some  $\varepsilon > 0$ . Then, for any  $\theta \in (0, \mathbb{E}[X])$ , we have*

$$(2.23) \quad \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_k(Z_n \leq e^{\theta n}) = \chi_k^*(\theta) \in (0, \infty),$$

where

$$(2.24) \quad \chi_k^*(\theta) = \sup_{\lambda \leq 0} \{\lambda\theta - \chi_k(\lambda)\} = \begin{cases} -r_k\theta - \log \gamma_k & \text{if } 0 < \theta < \theta_k, \\ \Lambda^*(\theta) & \text{if } \theta_k \leq \theta < \mathbb{E}[X], \end{cases}$$

with

$$(2.25) \quad \theta_k = \Lambda'(-r_k).$$

The value  $\chi_k^*(\theta)$  can be interpreted geometrically as the maximum distance between the graphs of the linear function  $l_\theta : \lambda \mapsto \theta\lambda$  with slope  $\theta$  and the function  $\chi_k : \lambda \mapsto \chi_k(\lambda)$  defined in (2.22). Taking into account the fact that  $\chi(\lambda) = \Lambda(\lambda)$  for  $\lambda \in [-r_k, 0]$  and  $\chi(\lambda) = \log \gamma_k$  for  $\lambda \leq -r_k$ , we can easily describe the phase transitions of  $\chi^*(\theta)$  depending on the value of the slope  $\theta$  of the function  $l_\theta$ :

- (1) in the case when  $\theta \in (\theta_k, \mathbb{E}[X])$ , the maximum  $\sup_{\lambda \leq 0} \{l_\theta(\lambda) - \chi(\lambda)\}$  is attained for  $\lambda \in (-r_k, 0)$  such that  $\chi'(\lambda_\theta) = \Lambda'(\lambda_\theta) = \theta$ , whose value is  $\chi^*(\theta) = \Lambda^*(\theta)$  (see Fig. 1);
- (2) the case when  $\theta = \theta_k$  is the critical slope for which the equation  $\Lambda'(\lambda) = \theta_k$  has a solution given by  $\lambda = -r_k$  (see Fig. 2);
- (3) in the case when  $\theta \in (0, \theta_k)$ , the maximum  $\sup_{\lambda \leq 0} \{l_\theta(\lambda) - \Lambda(\lambda)\}$  is attained for  $\lambda = -r_k$ , and then  $\chi^*(\theta) = -r_k\theta - \log \gamma_k$  becomes linear in  $\theta$  (see Fig. 3).

Fig. 1,2,3: Geometrical interpretation of  $\chi^*(\theta)$

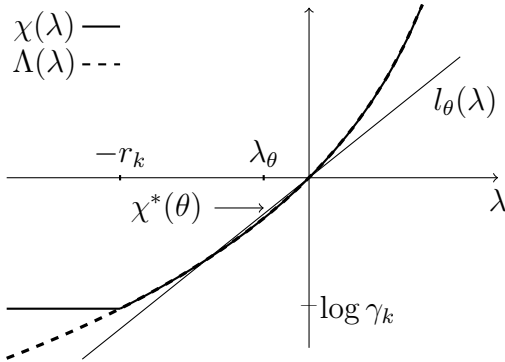


Fig. 1:  $\theta \in (\theta_k, \mathbb{E}[X])$

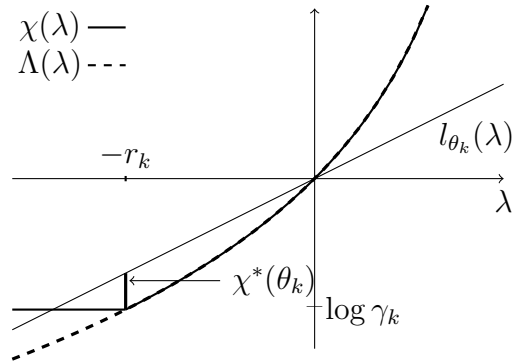
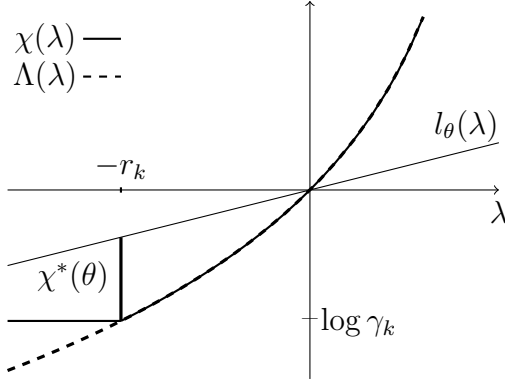


Fig. 2:  $\theta = \theta_k$

Fig. 3:  $\theta \in (0, \theta_k)$ 

**Remark 2.4.** *Theorem 2.3 corrects and improves the result of [8, Theorem 3.1(ii)]. Moreover it gives new and alternative expressions of the rate function and the critical value. Actually it was proved in [8, Theorem 3.1(ii)] that, assuming  $\mathbb{P}(Z_1 = 0) = 0$  and  $\mathbb{E}m_0^t < \infty$  for all  $t > 0$ , we have*

$$(2.26) \quad \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_k(Z_n \leq e^{\theta n}) = I_k(\theta) \in (0, \infty),$$

where

$$(2.27) \quad I_k(\theta) = \begin{cases} \rho_k \left(1 - \frac{\theta}{\theta_k^*}\right) + \frac{\theta}{\theta_k^*} \Lambda^*(\theta_k^*) & \text{if } 0 < \theta \leq \theta_k^*, \\ \Lambda^*(\theta) & \text{if } \theta_k^* \leq \theta < \mathbb{E}[X], \end{cases}$$

with

$$(2.28) \quad \rho_k = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_k(Z_n = j)$$

and  $\theta_k^*$  the unique solution on  $(0, \mathbb{E}[X])$  of the equation

$$(2.29) \quad \frac{\rho_k - \Lambda^*(\theta_k^*)}{\theta_k^*} = \inf_{0 \leq \theta \leq \mathbb{E}[X]} \frac{\rho_k - \Lambda^*(\theta)}{\theta}.$$

It has been stated mistakenly in [9] that  $\rho_k = -k \log \gamma$ , whereas the correct statement is

$$(2.30) \quad \rho_k = -\log \gamma_k,$$

according to [14, Theorem 2.3] (see Lemma 3.2 below). With this correction, the two critical values  $\theta_k$  and  $\theta_k^*$  and the two rate functions  $I_k$  and  $\chi_k^*$  coincide, that is

$$(2.31) \quad \theta_k = \theta_k^* \quad \text{and} \quad \chi_k^*(\theta) = I_k(\theta) \quad \text{for all } \theta \in (0, \mathbb{E}[X]).$$

Indeed, by the definition of  $\theta_k^*$ , the derivative of the function  $\theta \mapsto \frac{\rho_k - \Lambda^*(\theta)}{\theta}$  vanishes for  $\theta = \theta_k^*$ . Therefore, since  $(\Lambda^*)'(\theta) = \lambda_\theta$  with  $\Lambda'(\lambda_\theta) = \theta$ , we get, for  $\theta = \theta_k^*$ ,

$$(2.32) \quad \Lambda^*(\theta) = \lambda_\theta \theta + \rho_k.$$

Using the identity  $\Lambda^*(\theta) = \lambda_\theta \theta - \Lambda(\lambda_\theta)$ , we obtain

$$\Lambda(\lambda_\theta) = -\rho_k,$$

which implies that  $\lambda_\theta = -r_k$  and then  $\theta_k^* = \Lambda'(-r_k) = \theta_k$ .

Moreover, coming back to (2.32) and using the identities  $\Lambda(-r_k) = \log \gamma_k = -\rho_k$  and  $\theta_k = \Lambda'(-r_k)$ , we get

$$-r_k \theta_k - \Lambda(-r_k) = -r_k \theta_k + \rho_k = \Lambda_k^*(\theta_k).$$

Therefore, for any  $\theta \in [0, \theta_k]$ ,

$$\begin{aligned} -r_k \theta - \log \gamma_k &= -r_k \theta - \Lambda(-r_k) \\ &= \frac{\theta}{\theta_k} (-r_k \theta_k - \Lambda(-r_k)) + \frac{\theta}{\theta_k} \Lambda(-r_k) - \Lambda(-r_k) \\ &= \frac{\theta}{\theta_k} \Lambda_k^*(\theta_k) - \left(1 - \frac{\theta}{\theta_k}\right) \log \gamma_k, \end{aligned}$$

so that  $I_k(\theta) = \chi_k^*(\theta)$ , which ends the proof of (2.31). From (2.31) and Theorem 2.1, we see that (2.26) is valid assuming only  $\mathbb{P}(Z_1 = 0) = 0$  and  $\mathbb{E}[m_0^{r_k+\varepsilon}] < \infty$  for some  $\varepsilon > 0$ . Actually when  $\mathbb{P}(Z_1 = 0) > 0$ , as shown in [8, Theorem 3.1 (i)], (2.26) remains valid with  $\rho_k = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_k(Z_n = j) = \rho > 0$  independent of  $k$ .

Similarly, one can apply Theorem 2.1 to get the decay rate for the probability  $\mathbb{P}(Z_n \leq k_n)$ , where  $k_n$  is any sub-exponential sequence in the sense that  $k_n \rightarrow \infty$  and  $k_n / \exp(\theta n) \rightarrow 0$  for every  $\theta > 0$ , as stated in the following corollary.

**Corollary 2.5.** *Let  $k \geq 1$  and assume that  $\mathbb{E}[m_0^{r_k+\varepsilon}] < \infty$  for some  $\varepsilon > 0$ . Let  $k_n > 0$  be such that  $k_n \rightarrow \infty$  and  $k_n / \exp(\theta n) \rightarrow 0$  for every  $\theta > 0$ , as  $n \rightarrow \infty$ . Then*

$$(2.33) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_k(Z_n \leq k_n) = \log \gamma_k.$$

It was stated mistakenly in [8, Theorem 3.1(ii)] that  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_k(Z_n \leq k_n) = k \log \gamma$ . To show (2.33), it suffices to note that by Markov's inequality and Theorem 2.1, we have, for any  $r > r_k$ ,

$$\gamma_k^n = \mathbb{P}_k(Z_n = k) \leq \mathbb{P}_k(Z_n \leq k_n) \leq \mathbb{E}[Z_n^{-r}] k_n^r \leq \min\{\gamma_k^n k_n^r, \gamma_k^n k_n^{r_k} n\}.$$

The above argument leads to a precise large deviation bound as stated below.

**Corollary 2.6.** *Let  $k \geq 1$  and assume that  $\mathbb{E}[m_0^{r_k+\varepsilon}] < \infty$  for some  $\varepsilon > 0$ . Then*

$$(2.34) \quad \mathbb{P}_k(Z_n \leq e^{\theta n}) \leq \inf_{r>0} \frac{\mathbb{E}_k[Z_n^{-r}]}{e^{\theta r n}} = \begin{cases} e^{-n(-\theta r_k - \Lambda(-r_k))} & \text{if } 0 < \theta \leq \theta_k, \\ n e^{-n(-\theta_k r_k - \Lambda(-r_k))} & \text{if } \theta = \theta_k, \\ e^{-n\Lambda^*(\theta)} & \text{if } \theta_k \leq \theta < \mathbb{E}[X]. \end{cases}$$

The question of the exact decay rate of  $\mathbb{P}(Z_n \leq e^{\theta n})$  will be treated in a forthcoming paper.

As an example, let us consider the case where the reproduction law has a fractional linear generating function, that is when

$$(2.35) \quad p_0(\xi_0) = a_0 \quad \text{and} \quad p_k(\xi_0) = \frac{(1-a_0)(1-b_0)}{b_0} b_0^k \quad \text{for all } k \geq 1,$$

for which the generating function is

$$f_0(t) = a_0 + \frac{(1-a_0)(1-b_0)t}{1-b_0t},$$

where  $a_0 \in [0, 1)$  and  $b_0 \in (0, 1)$  are random variables depending on the environment  $\xi_0$ . This case has been examined by several authors (see e.g. [19, 22]). In the case where  $a_0 = 0$  (non-extinction), the BPRE is said to be geometric; in this case  $X = \log m_0 = -\log(1-b_0)$ ,  $\log \gamma_k = \log \mathbb{E}[e^{-kX}] = \Lambda(-k)$  and  $r_k = k$ . Therefore, we obtain the following explicit version of Theorem 2.1:

**Corollary 2.7.** *Let  $Z_n$  be a geometric BPRE. Assume that there exists  $\varepsilon > 0$  such that  $\mathbb{E}[e^{(k+\varepsilon)X}] < \infty$ . Then (2.23) holds with*

$$(2.36) \quad \chi_k^*(\theta) = \begin{cases} -k\theta - \log \mathbb{E}[e^{-kX}] & \text{if } 0 < \theta \leq \theta_k, \\ \Lambda^*(\theta) & \text{if } \theta_k \leq \theta < \mathbb{E}[X], \end{cases}$$

where

$$(2.37) \quad \theta_k = \mathbb{E}[Xe^{-kX}]/\mathbb{E}[e^{-kX}].$$

Corollary 2.7 recovers and completes the large deviation result in [8, Corollary 3.3] for a fractional linear BPRE with  $a_0 > 0$ , where (2.23) was obtained with

$$(2.38) \quad I(\theta) = \begin{cases} -\theta - \log \mathbb{E}[e^{-X}] & \text{if } 0 < \theta \leq \theta^*, \\ \Lambda^*(\theta) & \text{if } \theta^* \leq \theta < \mathbb{E}[X], \end{cases}$$

and

$$(2.39) \quad \theta^* = \mathbb{E}[Xe^{-X}]/\mathbb{E}[e^{-X}].$$

In fact the result in [8, Corollary 3.3] was stated without the hypothesis  $a_0 > 0$ , but when  $a_0 = 0$ , this result is valid only for  $k = 1$ , as shown by Corollary 2.7 (for  $k \geq 2$ , the factor  $k$  is missing in [8, Corollary 3.3]).

As another consequence of Theorem 2.1, we improve an earlier result about the rate of convergence in the central limit theorem for  $W - W_n$ .

**Theorem 2.8.** *Assume  $\text{essinf } \frac{m_0(2)}{m_0^2} > 1$  and  $\mathbb{E}Z_1^{2+\varepsilon} < \infty$  for some  $\varepsilon \in (0, 1]$ . Then there exists a constant  $C > 0$  such that, for all  $k \geq 1$ ,*

$$(2.40) \quad \sup_{x \in \mathbb{R}} \left| \mathbb{P}_k \left( \frac{\Pi_n(W - W_n)}{\sqrt{Z_n} \delta_\infty(T^n \xi)} \leq x \right) - \Phi(x) \right| \leq CA_{k,n}(\varepsilon/2),$$

where  $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$  is the standard normal distribution function.

Theorem 2.8 improves the exponential rate of convergence in [17, Theorem 1.7]. In our approach the assumption  $\text{essinf} \frac{m_0(2)}{m_0^2} > 1$  is required to ensure that the quenched variance  $\delta_\infty(\xi)$  of the variable  $W$  is a.s. separated from 0. This hypothesis does not seem natural and should be relaxed. One should be able to find a suitable hypothesis to ensure the existence of harmonic moments for the random variable  $\delta_\infty(\xi)$ , which would be enough for our objective.

As another consequence of Theorem 2.1, we give some large deviation results on the ratio

$$(2.41) \quad R_n = \frac{Z_{n+1}}{Z_n}$$

toward the conditional mean  $m_n = \sum_{k=1}^{\infty} k p_k(\xi_n)$ . Let

$$(2.42) \quad M_{n,j} = j^{-1} \sum_{i=1}^j N_{n,i}$$

be the empirical mean of  $m_n$  of size  $j$  under the environment  $\xi$ , where the r.v.'s  $N_{n,i}$  ( $i = 1, \dots, j$ ) are i.i.d. with generating function  $f_n$ .

**Theorem 2.9.** *Let  $k \geq 1$ . If, for some set  $D \subset \mathbb{R}$ , there exist some constants  $C_1 > 0$  and  $r > 0$  such that, for all  $j \geq 1$ ,*

$$(2.43) \quad \mathbb{P}(M_{0,j} - m_0 \in D) \leq \frac{C_1}{j^r},$$

*then there exists a constant  $B_1 \in (0, \infty)$  such that for all  $n \geq 1$ ,*

$$(2.44) \quad \mathbb{P}_k(R_n - m_n \in D) \leq B_1 A_{k,n}(r),$$

*where  $A_{k,n}(r)$  is defined in (2.16). Similarly, if there exist some constants  $C_2 > 0$  and  $r > 0$  such that, for all  $j \geq 1$ ,*

$$(2.45) \quad \mathbb{P}(M_{0,j} - m_0 \in D) \geq \frac{C_2}{j^r},$$

*then there exists a constant  $B_2 \in (0, \infty)$  such that for all  $n \geq 1$ ,*

$$(2.46) \quad \mathbb{P}_k(R_n - m_n \in D) \geq B_2 A_{k,n}(r).$$

This result shows that there exist some phase transitions in the rate of convergence depending on whether the value of  $r$  is less than, equal or greater than the constant  $\gamma_k$ . The next result gives a bound of the large deviation probability of  $R_n - m_n$  under a simple moment condition on  $Z_1$ .

**Theorem 2.10.** *Let  $k \geq 1$ . Assume that there exists  $p > 1$  such that  $\mathbb{E}|Z_1 - m_0|^p < \infty$ . Then, there exists a constant  $C_p > 0$  such that, for any  $a > 0$ ,*

$$(2.47) \quad \mathbb{P}_k(|R_n - m_n| > a) \leq \begin{cases} C_p a^{-p} A_{k,n}(p-1) & \text{if } p \in (1, 2], \\ C_p a^{-p} A_{k,n}(p/2) & \text{if } p \in (2, \infty). \end{cases}$$

### 3. PROOF OF MAIN THEOREMS

In this section we will prove the main results of this paper, Theorems 2.1 and 2.3, and the associated result, Proposition 2.2. In Section 3.1 we present some auxiliary results concerning the critical value for the existence of the harmonic moments of the r.v.  $W$  and the asymptotic behavior of the asymptotic distribution  $\mathbb{P}_k(Z_n = j)$  as  $n \rightarrow \infty$ , with  $j \geq k \geq 1$ . In Sections 3.2 and 3.3 we prove respectively Theorems 2.1 and 2.3. The proof of Proposition 2.2 is given in Section 3.4 for a Galton-Watson process and in Section 3.5 for a general BPRE.

**3.1. Auxiliary results.** We recall some results to be used in the proofs. The first one concerns the critical value for the existence of harmonic moments of the r.v.  $W$ .

**Lemma 3.1** ([14], Theorem 2.1). *Assume that there exists a constant  $p > 0$  such that  $\mathbb{E}[m_0^p] < \infty$ . Then for any  $a \in (0, p)$ ,*

$$\mathbb{E}_k W^{-a} < \infty \quad \text{if and only if} \quad \mathbb{E}[p_1^k(\xi_0)m_0^a] < 1.$$

The second result is about the asymptotic equivalent of the probability  $\mathbb{P}_k(Z_n = j)$  as  $n \rightarrow \infty$ , for any  $j \geq k \geq 1$ .

**Lemma 3.2** ([14], Theorem 2.3). *Assume that  $\mathbb{P}(Z_1 = 1) > 0$ . For any  $k \geq 1$  the following assertions hold.*

a) *For any accessible state  $j \geq k$  in the sense that  $\mathbb{P}_k(Z_l = j) > 0$  for some  $l \geq 0$ , we have*

$$(3.1) \quad \mathbb{P}_k(Z_n = j) \underset{n \rightarrow \infty}{\sim} \gamma_k^n q_{k,j},$$

where  $q_{k,k} = 1$  and, for  $j > k$ ,  $q_{k,j} \in (0, +\infty)$  is the solution of the recurrence relation

$$(3.2) \quad \gamma_k q_{k,j} = \sum_{i=k}^j p(i, j) q_{k,i},$$

with  $q_{k,i} = 0$  for any non-accessible state  $i$  (i.e.  $\mathbb{P}_k(Z_l = i) = 0$  for all  $l \geq 0$ ).

b) *Assume that there exists  $\varepsilon > 0$  such that  $\mathbb{E}[m_0^{r_k + \varepsilon}] < \infty$ . Then, for any  $r > r_k$ , we have*

$$(3.3) \quad \sum_{j=k}^{\infty} j^{-r} q_{k,j} < \infty.$$

*In particular, the radius of convergence of the power series*

$$(3.4) \quad Q_k(t) = \sum_{j=k}^{+\infty} q_{k,j} t^j$$

*is equal to 1.*

c) *For all  $t \in [0, 1)$  and  $k \geq 1$ , we have,*

$$(3.5) \quad \frac{G_{k,n}(t)}{\gamma_k^n} \uparrow Q_k(t) \quad \text{as } n \rightarrow \infty,$$

where  $G_{k,n}$  is the probability generating function of  $Z_n$  when  $Z_0 = k$ , defined in (2.4).

d)  $Q_k(t)$  is the unique power series which verifies the functional equation

$$(3.6) \quad \gamma_k Q_k(t) = \mathbb{E}[Q_k(f_0(t))], \quad t \in [0, 1),$$

with the condition  $Q_k^{(k)}(0) = 1$ .

**3.2. Proof of Theorem 2.1.** In this section we give a proof of the convergence of the normalized harmonic moments  $\mathbb{E}_k[Z_n^{-r}]/A_{k,n}(r)$  as  $n \rightarrow \infty$ , where  $A_{k,n}(r)$  is defined in (2.16).

a) We first consider the case when  $r < r_k$  (which corresponds to the case  $\gamma_k < c_r$ ). By the change of measure (2.14), we obtain

$$(3.7) \quad \mathbb{E}_k[Z_n^{-r}] = \mathbb{E}_k^{(r)}[W_n^{-r}]c_r^n,$$

with  $c_r = \mathbb{E}m_0^{-r}$ . From (2.15) and [17, Lemma 2.1], it follows that the sequence  $(\mathbb{E}_k^{(r)}[W_n^{-r}])$  is increasing and

$$(3.8) \quad \lim_{n \rightarrow \infty} \mathbb{E}_k^{(r)}[W_n^{-r}] = \sup_{n \in \mathbb{N}} \mathbb{E}_k^{(r)}[W_n^{-r}] = \mathbb{E}_k^{(r)}[W^{-r}].$$

Moreover, for any  $r < r_k$ , we have  $\gamma_k < c_r$ , which implies that  $\mathbb{E}^{(r)}[p_1^k(\xi_0)m_0^r] = \gamma_k/\mathbb{E}m_0^{-r} < 1$ . So, by Lemma 3.1, we get, for any  $r < r_k$ ,

$$(3.9) \quad \mathbb{E}_k^{(r)}[W^{-r}] < \infty.$$

Therefore, coming back to (3.7) and using (3.8) and (3.9), we obtain

$$(3.10) \quad \mathbb{E}_k[Z_n^{-r}]c_r^{-n} \underset{n \rightarrow \infty}{\uparrow} \mathbb{E}_k^{(r)}[W^{-r}] \in (0, \infty).$$

To give an integral expression of the limit constant  $C(k, r)$ , we shall use the following expression for the inverse of a positive random variable  $X^r$ : for any  $r > 0$ , we have

$$(3.11) \quad \frac{1}{X^r} = \frac{1}{\Gamma(r)} \int_0^{+\infty} e^{-uX} u^{r-1} du.$$

Then, from (3.10), (3.11) and Fubini's theorem, we get

$$(3.12) \quad \mathbb{E}_k[Z_n^{-r}]c_r^{-n} \underset{n \rightarrow \infty}{\uparrow} \frac{1}{\Gamma(r)} \int_0^\infty \phi_k^{(r)}(u) u^{r-1} du,$$

which proves (2.17) for  $r < r_k$ .

b) Next we consider the case when  $r > r_k$  (which corresponds to the case  $\gamma_k > c_r$ ). Using parts a) and b) of Lemma 3.2 and the monotone convergence theorem, it follows that

$$(3.13) \quad \begin{aligned} \lim_{n \rightarrow \infty} \uparrow \frac{\mathbb{E}_k[Z_n^{-r}]}{\gamma_k^n} &= \lim_{n \rightarrow \infty} \uparrow \sum_{j=k+1}^\infty k^{-r} \frac{\mathbb{P}(Z_n = k)}{\gamma_k^n} \\ &= \sum_{j=k+1}^\infty k^{-r} q_{k,j} < \infty. \end{aligned}$$



Moreover, using (3.11) together with Fubini's theorem and the change of variable  $ju = t$ , we obtain

$$\frac{1}{\Gamma(r)} \int_0^1 Q_k(e^{-u}) u^{r-1} du = \sum_{j=k}^{\infty} q_{k,j} j^{-r} \frac{1}{\Gamma(r)} \int_0^{\infty} e^{-t} t^{r-1} dt = \sum_{j=k}^{\infty} q_{k,j} j^{-r}.$$

Therefore, coming back to (3.13), we get

$$(3.14) \quad \lim_{n \rightarrow \infty} \uparrow \frac{\mathbb{E}_k[Z_n^{-r}]}{\gamma_k^n} = \frac{1}{\Gamma(r)} \int_0^1 Q_k(e^{-u}) u^{r-1} du,$$

which proves (2.17) for  $r > r_k$ .

c) Now we consider the case when  $r = r_k$  (which corresponds to  $r = \gamma_k$ ). For  $n \geq 1$  and  $m \geq 0$ , we have the following well-known branching property for  $Z_n$ :

$$(3.15) \quad Z_{n+m} = \sum_{i=1}^{Z_m} Z_{n,i}^{(m)},$$

where the r.v.'s  $Z_{n,i}^{(m)}$  ( $i \geq 1$ ) are independent of  $Z_m$  under  $\mathbb{P}_\xi$  and  $\mathbb{P}$ . Moreover, under  $\mathbb{P}_\xi$ , for each  $n \geq 0$ , the r.v.'s  $Z_{n,i}^{(m)}$  ( $i \geq 1$ ) are i.i.d. with the same conditional probability law  $\mathbb{P}_\xi(Z_{n,i}^{(m)} \in \cdot) = \mathbb{P}_{T^m \xi}(Z_n \in \cdot)$ , where  $T^m$  is the shift operator defined by  $T^m(\xi_0, \xi_1, \dots) = (\xi_m, \xi_{m+1}, \dots)$ . Intuitively, relation (3.15) shows that, conditionally on  $Z_m = i$ , the annealed law of the process  $Z_{n+m}$  is the same as that of a new process  $Z_n$  starting with  $i$  individuals. Using (3.15) with  $m = 1$ , we obtain

$$(3.16) \quad \mathbb{E}_k[Z_{n+1}^{-r}] = \mathbb{E}_k[Z_n^{-r}] \mathbb{P}_k(Z_1 = k) + \sum_{i=k+1}^{\infty} \mathbb{E}_i[Z_n^{-r}] \mathbb{P}_k(Z_1 = i).$$

From (3.7), we have  $\mathbb{E}_i[Z_n^{-r}] = \mathbb{E}_i^{(r)}[W_n^{-r}] c_r^n$ . Substituting this into (3.16) and setting

$$b_n = \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W_n^{-r}] \mathbb{P}_k(Z_1 = i),$$

we get

$$(3.17) \quad \mathbb{E}_k[Z_{n+1}^{-r}] = \mathbb{E}_k[Z_n^{-r}] \gamma_k + b_n c_r^n,$$

with  $\gamma_k = \mathbb{P}_k(Z_1 = k)$ . Iterating (3.17) leads to

$$(3.18) \quad \mathbb{E}_k[Z_{n+1}^{-r}] = \gamma_k^{n+1} k^{-r} + \sum_{j=0}^n \gamma_k^{n-j} b_j c_r^j.$$

Using the fact that  $r = r_k$  (which corresponds to  $\gamma_k = c_r$ ) and dividing (3.18) by  $\gamma_k^{n+1} n$ , we get

$$(3.19) \quad \frac{\mathbb{E}_k[Z_{n+1}^{-r}]}{n \gamma_k^{n+1}} = \frac{k^{-r}}{n} + \frac{\gamma_k^{-1}}{n} \sum_{j=0}^n b_j.$$

To prove the convergence in (3.19) we need to show that  $\lim_{n \rightarrow \infty} b_n < \infty$ . By (3.8) and the monotone convergence theorem, we have

$$(3.20) \quad b := \lim_{n \rightarrow \infty} \uparrow b_n = \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}] \mathbb{P}_k(Z_1 = i).$$

Now we show that  $b < \infty$ . Using (3.15) for  $m = 0$  and  $Z_0 = i$ , with  $i > k$ , and the fact that  $Z_{n,j} > 0$  for all  $1 \leq j \leq i$ , we obtain

$$(3.21) \quad \mathbb{E}_i[Z_n^{-r}] = \mathbb{E}[(Z_{n,1} + \dots + Z_{n,i})^{-r}] \leq \mathbb{E}[(Z_{n,1} + \dots + Z_{n,k})^{-r}] = \mathbb{E}_k[Z_n^{-r}].$$

By (3.21) and the change of measure (3.7), we get, for any  $i \geq k+1$ ,

$$\mathbb{E}_i^{(r)}[W_n^{-r}] \leq \mathbb{E}_{k+1}^{(r)}[W_n^{-r}].$$

Then, as in (3.8), letting  $n \rightarrow \infty$  leads to

$$(3.22) \quad \mathbb{E}_i^{(r)}[W^{-r}] \leq \mathbb{E}_{k+1}^{(r)}[W^{-r}].$$

Now we shall prove that

$$(3.23) \quad \mathbb{E}_{k+1}^{(r)}[W^{-r}] < \infty.$$

For this it is enough to verify the condition of Lemma 3.1 under the measure  $\mathbb{E}_k^{(r)}$  defined by (2.14): indeed, for  $r = r_k$  we have  $\gamma_k = c_r$ , which implies that

$$\mathbb{E}^{(r)}[p_1^{k+1}(\xi_0)m_0^r] = \frac{\gamma_{k+1}}{\mathbb{E}m_0^{-r}} = \frac{\gamma_{k+1}}{\gamma_k} < 1.$$

This proves (3.23). Using (3.22) and (3.23), we obtain

$$b = \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}] \mathbb{P}_k(Z_1 = i) \leq \mathbb{E}_{k+1}^{(r)}[W^{-r}] \sum_{j=k+1}^{\infty} \mathbb{P}_k(Z_1 = j) < \infty.$$

Therefore, coming back to (3.19), using (3.20) and Cesaro's lemma, we get

$$(3.24) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}_k[Z_{n+1}^{-r}]}{n\gamma_k^{n+1}} = \frac{1}{\gamma_k} \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}] \mathbb{P}_k(Z_1 = i) < \infty,$$

which proves (2.17) for  $r = r_k$ , with

$$C(k, r) = \frac{1}{\gamma_k} \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}] \mathbb{P}_k(Z_1 = i).$$

We now show an integral expression of the constant  $C(k, r)$ . Recall that  $W$  admits the following decomposition

$$(3.25) \quad W = \sum_{j=1}^{Z_0} W(j),$$

where the r.v.'s  $W(j)$  ( $j = 1, 2, \dots$ ) are independent of  $Z_0$  and  $m_0$  under  $\mathbb{P}_\xi$  and  $\mathbb{P}$ . Moreover, conditionally on the environnement  $\xi$ , the r.v.'s  $W(j)$  ( $j = 1, 2, \dots$ ) are

i.i.d. with common law  $\mathbb{P}_\xi(W(j) \in \cdot) = \mathbb{P}_\xi(W \in \cdot)$ . With these considerations, it can be easily seen that

$$(3.26) \quad \phi_i^{(r)}(t) = \mathbb{E}_i^{(r)}[\phi_\xi(u)] = \mathbb{E}^{(r)}[\phi_\xi(u)^i].$$

Therefore, using (3.11) with  $r = r_k$ , together with (3.26) and Fubini's theorem, we obtain

$$\begin{aligned} \frac{1}{\gamma_k} \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}] \mathbb{P}_k(Z_1 = i) &= \frac{1}{\gamma_k \Gamma(r)} \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)} \int_0^\infty e^{-uW} u^{r-1} du \mathbb{P}_k(Z_1 = i) \\ &= \frac{1}{\gamma_k \Gamma(r)} \sum_{i=k+1}^{\infty} \mathbb{E}^{(r)} \int_0^\infty \phi_\xi^i(u) u^{r-1} du \mathbb{P}_k(Z_1 = i) \\ &= \frac{1}{\gamma_k \Gamma(r)} \mathbb{E}^{(r)} \int_0^\infty \sum_{i=k+1}^{\infty} \phi_\xi^i(u) \mathbb{P}_k(Z_1 = i) u^{r-1} du \\ (3.27) \quad &= \frac{1}{\gamma_k \Gamma(r)} \mathbb{E}^{(r)} \left[ \int_0^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right], \end{aligned}$$

where  $\bar{G}_{k,1}(u) = G_{k,1}(u) - \gamma_k u^k = \sum_{j=k+1}^{\infty} u^j \mathbb{P}(Z_1 = j)$ . Therefore, using (3.24) and (3.27), we get

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_k[Z_{n+1}^{-r}]}{n \gamma_k^{n+1}} = \frac{1}{\gamma_k \Gamma(r)} \mathbb{E}^{(r)} \left[ \int_0^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right],$$

which ends the proof of Theorem 2.1.

**3.3. Proof of Theorem 2.3.** In this section we prove Theorem 2.3. For convenience, let  $\lambda_k = -r_k$ . From Theorem 2.1, for any  $k \geq 1$ , we have

$$(3.28) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_k[Z_n^\lambda] = \chi_k(\lambda) = \begin{cases} \log \gamma_k & \text{if } \lambda \leq \lambda_k, \\ \Lambda(\lambda) & \text{if } \lambda \in [\lambda_k, 0]. \end{cases}$$

Thus using a version of the Gärtner-Ellis theorem adapted to the study of tail probabilities (see [17, Lemma 3.1]) and the fact that  $\chi_k(\lambda) = \log \gamma_k$  for all  $\lambda \leq \lambda_k$ , we obtain, for all  $\theta \in (0, \mathbb{E}[X])$ ,

$$(3.29) \quad \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_k(Z_n \leq e^{\theta n}) = \chi^*(\theta),$$

with

$$(3.30) \quad \chi_k^*(\theta) = \sup_{\lambda \leq 0} \{\lambda \theta - \chi_k(\lambda)\} = \max \left\{ \lambda_k \theta - \Lambda(\lambda_k), \sup_{\lambda_k \leq \lambda \leq 0} \{\lambda \theta - \Lambda(\lambda)\} \right\}.$$

It is well-known (see e.g. [13, Lemma 2.2.5]) that the function

$$\Lambda^*(\theta) = \sup_{\lambda \leq 0} \{\lambda \theta - \Lambda(\lambda)\} = \{\lambda_\theta \theta - \Lambda(\lambda_\theta)\}, \quad \text{with } \Lambda'(\lambda_\theta) = \theta,$$

is non-increasing for  $\theta \in (0, \mathbb{E}[X])$ . Therefore, letting

$$(3.31) \quad \theta_k = \Lambda'(\lambda_k),$$

it follows that:

(1) for any  $\theta \in (0, \theta_k]$ ,

$$\lambda_k \theta - \Lambda(\lambda_k) \geq \lambda_k \theta_k - \Lambda(\lambda_k) = \Lambda^*(\theta_k) = \sup_{\lambda_k \leq \lambda \leq 0} \{\lambda \theta - \Lambda(\lambda)\};$$

(2) for any  $\theta \in [\theta_k, \mu)$ ,

$$\Lambda^*(\theta) = \sup_{\lambda_k \leq \lambda \leq 0} \{\lambda \theta - \Lambda(\lambda)\} \geq \Lambda^*(\theta_k) = \lambda_k \theta_k - \Lambda(\lambda_k) \geq \lambda_k \theta - \Lambda(\lambda_k).$$

With these considerations, we get from (3.30) that

$$(3.32) \quad \chi_k^*(\theta) = \begin{cases} \lambda_k \theta - \Lambda(\lambda_k) & \text{if } \theta \in (0, \theta_k], \\ \Lambda^*(\theta) & \text{if } \theta \in [\theta_k, \mathbb{E}[X]], \end{cases}$$

which ends the proof of Theorem 2.3.

**3.4. Proof of Proposition 2.2 for the Galton-Watson case.** In this section we assume that  $(Z_n)$  is a Galton-Watson process and prove (1.6), which is a particular but simpler case of Proposition 2.2.

*Proof.* First note that for the Galton-Watson case, we have

$$(3.33) \quad \gamma m_1^r = 1.$$

For convenience, we shall write  $r = r_1$ . Using the additive property of integration and the change of variable  $u = tm^k$  for  $k \geq 0$ , together with Fubini's theorem and the fact that  $\gamma m^r = 1$ , we obtain

$$(3.34) \quad \begin{aligned} \frac{1}{\gamma} \int_1^\infty \bar{G}(\phi(u)) u^{r-1} du &= \frac{1}{\gamma} \sum_{k=0}^\infty \int_{m^k}^{m^{k+1}} \bar{G}(\phi(u)) u^{r-1} du \\ &= \frac{1}{\gamma} \sum_{k=0}^\infty \int_1^m \bar{G}(\phi(tm^k)) (m^r)^k t^{r-1} dt \\ &= \frac{1}{\gamma} \int_1^m \sum_{k=0}^\infty \gamma^{-k} \bar{G}(\phi(tm^k)) t^{r-1} dt. \end{aligned}$$

Since  $\bar{G}(t) = G(t) - \gamma t$  and  $G(\phi(t)) = \phi(tm)$ , we obtain, for any  $k \geq 0$ ,

$$(3.35) \quad \begin{aligned} \gamma^{-k} \bar{G}(\phi(tm^k)) &= \gamma^{-k} G(\phi(tm^k)) - \gamma^{-k} \gamma \phi(tm^k) \\ &= \gamma^{-k} G^{\circ k+1}(\phi(t)) - \gamma^{k-1} G^{\circ k}(\phi(t)). \end{aligned}$$

By (3.35), using a telescoping argument and the fact that  $\lim_{k \rightarrow \infty} \gamma^{-k} G^{\circ k}(t) = Q(t)$ , we get

$$(3.36) \quad \sum_{k=0}^\infty \gamma^{-k} \bar{G}(\phi(tm^k)) = \gamma Q(\phi(t)) - \gamma \phi(t).$$

Therefore, coming back to (3.34) and using (3.36), we have

$$(3.37) \quad \frac{1}{\gamma} \int_1^\infty \bar{G}(\phi(u)) u^{r-1} du = \int_1^m Q(\phi(u)) u^{r-1} du - \int_1^m \phi(u) u^{r-1} du.$$

Moreover, using the change of variable  $u = t/m$  and the relations  $G(\phi(t/m)) = \phi(t)$  and  $\gamma m^r = 1$ , we get

$$\frac{1}{\gamma} \int_0^1 G(\phi(u)) u^{r-1} du = \frac{m^{-r}}{\gamma} \int_0^m \phi(t) t^{r-1} dt = \int_0^m \phi(t) t^{r-1} dt.$$

Therefore, since  $\bar{G}(u) = G(u) - \gamma u$ , we obtain

$$\begin{aligned} \frac{1}{\gamma} \int_0^1 \bar{G}(\phi(u)) u^{r-1} du &= \frac{1}{\gamma} \int_0^1 G(\phi(u)) u^{r-1} du - \int_0^1 \phi(u) u^{r-1} du \\ &= \int_0^m \phi(u) u^{r-1} du - \int_0^1 \phi(u) u^{r-1} du \\ (3.38) \quad &= \int_1^m \phi(u) u^{r-1} du. \end{aligned}$$

Finally, using (3.37), (3.38) and the additive property of integration, we obtain

$$(3.39) \quad \frac{1}{\gamma} \int_0^\infty \bar{G}(\phi(u)) u^{r-1} du = \int_1^m Q(\phi(u)) u^{r-1} du,$$

which ends the proof of (1.6).  $\square$

**3.5. Proof of Proposition 2.2.** Let  $k \geq 1$ . For convenience, let  $r = r_k$ . Using the additive property of integration, the change of variable  $u = t\Pi_j^k$  for  $j \geq 0$  and Fubini's theorem, we have

$$\begin{aligned} \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \int_1^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right] &= \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \sum_{j=0}^\infty \int_{\Pi_j}^{\Pi_{j+1}} \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right] \\ &= \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \sum_{j=0}^\infty \int_1^{m_j} \bar{G}_{k,1}(\phi_\xi(t\Pi_j)) \Pi_j^r t^{r-1} dt \right] \\ (3.40) \quad &= \frac{1}{\gamma_k} \sum_{j=0}^\infty \mathbb{E}^{(r)} \left[ \int_1^{m_j} \bar{G}_{k,1}(\phi_\xi(t\Pi_j)) \Pi_j^r t^{r-1} dt \right]. \end{aligned}$$

Recall that  $\phi_\xi(t\Pi_j) = g_j(\phi_{T^j\xi}(t))$ , where  $g_j(t) = f_0 \circ \dots \circ f_{j-1}(t)$  is a random function depending on the environment  $\xi_0, \dots, \xi_{j-1}$  and  $T^j\xi = (\xi_j, \xi_{j+1}, \dots)$ . Then, using the change of measure (2.14), the independence of the environment sequence  $(\xi_i)$  and Fubini's theorem, we see that

$$(3.41) \quad \mathbb{E}^{(r)} \left[ \int_1^{m_j} \bar{G}_{k,1}(\phi_\xi(t\Pi_j)) \Pi_j^r t^{r-1} dt \right] = \mathbb{E}^{(r)} \left[ \int_1^{m_j} c_r^{-j} \mathbb{E}_{T^j\xi} \left[ \bar{G}_{k,1}(g_j(\phi_{T^j\xi}(t))) \right] t^{r-1} dt \right].$$

Using the fact that  $\bar{G}_{k,1}(t) = G_{k,1}(t) - \gamma_k t^k$  and the relations  $\mathbb{E}[G_{k,n}(g_j(t))] = G_{k,n+j}(t)$  and  $\mathbb{E}[g_j^k(t)] = G_{k,j}(t)$ , we get

$$\begin{aligned} &\mathbb{E}^{(r)} \left[ \int_1^{m_j} c_r^{-j} \mathbb{E}_{T^j\xi} \left[ \bar{G}_{k,1}(g_j(\phi_{T^j\xi}(t))) \right] t^{r-1} dt \right] \\ (3.42) \quad &= \mathbb{E}^{(r)} \left[ \int_1^{m_j} c_r^{-j} \left[ G_{k,j+1}(\phi_{T^j\xi}(t)) - G_{k,j}(\phi_{T^j\xi}(t)) \right] t^{r-1} dt \right]. \end{aligned}$$

Moreover, since the environment sequence  $(\xi_0, \xi_1, \dots)$  is i.i.d., we obtain, for any  $j \geq 0$ ,

$$(3.43) \quad \begin{aligned} & \mathbb{E}^{(r)} \left[ \int_1^{m_j} c_r^{-j} \left[ G_{k,j+1}(\phi_{T^j \xi}(t)) - G_{k,j}(\phi_{T^j \xi}(t)) \right] t^{r-1} dt \right] \\ &= \mathbb{E}^{(r)} \left[ \int_1^{m_0} c_r^{-j} \left[ G_{k,j+1}(\phi_\xi(t)) - G_{k,j}(\phi_\xi(t)) \right] t^{r-1} dt \right]. \end{aligned}$$

Therefore, coming back to (3.40) and using the fact that  $c_r = \gamma_k$  (for  $r = r_k$ ) together with Fubini's theorem, we obtain

$$(3.44) \quad \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \int_1^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right] = \mathbb{E}^{(r)} \left[ \int_1^{m_0} \sum_{j=0}^\infty \frac{G_{k,j+1}(\phi_\xi(t))}{\gamma_k^{j+1}} - \frac{G_{k,j}(\phi_\xi(t))}{\gamma_k^j} t^{r-1} dt \right].$$

Using a telescoping argument and the assertion that  $\lim_{j \rightarrow \infty} \gamma_k^j G_{k,j}(t) = Q_k(t)$  for all  $t \in [0, 1)$ , we get

$$(3.45) \quad \begin{aligned} \sum_{j=0}^\infty \left[ \frac{G_{k,j+1}(\phi_\xi(t))}{\gamma_k^{j+1}} - \frac{G_{k,j}(\phi_\xi(t))}{\gamma_k^j} \right] &= Q_k(\phi_\xi(t)) - G_{k,0}(\phi_\xi(t)) \\ &= Q_k(\phi_\xi(t)) - \phi_\xi^k(t). \end{aligned}$$

Therefore, by (3.44) and (3.45), we have

$$(3.46) \quad \begin{aligned} & \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \int_1^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right] \\ &= \mathbb{E}^{(r)} \left[ \int_1^{m_0} Q_k(\phi_\xi(t)) t^{r-1} dt \right] - \mathbb{E}^{(r)} \left[ \int_1^{m_0} \phi_\xi^k(t) t^{r-1} dt \right]. \end{aligned}$$

Moreover, using the identity  $\phi_\xi(u) = f_0(\phi_{T\xi}(t/m_0))$ , the change of variable  $t = u/m_0$ , the independence between  $\xi_0$  and  $T\xi$ , the relation  $\gamma_k = c_r$  and Fubini's theorem, we get

$$(3.47) \quad \begin{aligned} \mathbb{E}^{(r)} \left[ \int_0^{m_0} \phi_\xi^k(t) t^{r-1} dt \right] &= \mathbb{E}^{(r)} \left[ \int_0^1 f_0^k(\phi_{T\xi}(u)) m_0^r u^{r-1} du \right] \\ &= \mathbb{E}^{(r)} \left[ \int_0^1 \mathbb{E}_{T\xi}^{(r)} \left[ f_0^k(\phi_{T\xi}(u)) m_0^r \right] u^{r-1} du \right] \\ &= \mathbb{E}^{(r)} \left[ \int_0^1 c_r^{-1} G_{k,1}(\phi_{T\xi}(u)) u^{r-1} du \right] \\ &= \gamma_k^{-1} \mathbb{E}^{(r)} \left[ \int_0^1 G_{k,1}(\phi_\xi(u)) u^{r-1} du \right]. \end{aligned}$$

Therefore, from the identity  $\bar{G}_{k,1}(t) = G_{k,1}(t) - \gamma_k t^k$  and (3.47), it follows that

$$(3.48) \quad \begin{aligned} \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \int_0^1 \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right] &= \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \int_0^1 G_{k,1}(\phi_\xi(t)) t^{r-1} dt \right] - \mathbb{E}^{(r)} \left[ \int_0^1 \phi_\xi^k(t) t^{r-1} dt \right] \\ &= \mathbb{E}^{(r)} \left[ \int_0^{m_0} \phi_\xi^k(t) t^{r-1} dt \right] - \mathbb{E}^{(r)} \left[ \int_0^1 \phi_\xi^k(t) t^{r-1} dt \right] \\ &= \mathbb{E}^{(r)} \left[ \int_1^{m_0} \phi_\xi^k(t) t^{r-1} dt \right]. \end{aligned}$$

Finally, using (3.46) and (3.48), we obtain

$$(3.49) \quad \frac{1}{\gamma_k} \mathbb{E}^{(r)} \left[ \int_0^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du \right] = \mathbb{E}^{(r)} \left[ \int_1^{m_0} Q_k(\phi_\xi(u)) u^{r-1} du \right],$$

which ends the proof of Proposition 2.2.

#### 4. APPLICATIONS

In this section we present the proofs of Theorems 2.8, 2.9 and 2.10 as applications of Theorem 2.1. In Section 4.1 we give the rate of convergence in the central limit theorem for  $W - W_n$  where we prove Theorem 2.8. In Section 4.2 we deal with the large deviation results for the ratio  $R_n = Z_{n+1}/Z_n$ , where we prove Theorems 2.9 and 2.10.

**4.1. Central Limit Theorem for  $W - W_n$ .** In this section we prove Theorem 2.8.

*Proof of Theorem 2.8.* It is well known that  $W$  admits the following decomposition:

$$\Pi_n(W - W_n) = \sum_{i=1}^{Z_n} (W(i) - 1),$$

where under  $\mathbb{P}_\xi$ , the random variables  $W(i)$  ( $i \geq 1$ ) are independent of each other and independent of  $Z_n$ , with common distribution  $\mathbb{P}_\xi(W(i) \in \cdot) = \mathbb{P}_{T^n \xi}(W \in \cdot)$ . Let

$$\delta_\infty^2(\xi) = \sum_{n=0}^{\infty} \frac{1}{\Pi_n} \left( \frac{m_n(2)}{m_n^2} - 1 \right).$$

The r.v.  $\delta_\infty^2(\xi)$  is the variance of  $W$  under  $\mathbb{P}_\xi$  (see e.g. [16]). Notice that if  $c_0 := \text{essinf } \frac{m_0(2)}{m_0^2} > 1$ , then  $\delta_\infty^2(\xi) \geq c_0 - 1 > 0$ . Therefore, condition  $\mathbb{E} Z_1^{2+\varepsilon} < \infty$  implies that, for all  $k \geq 1$ , it holds  $\mathbb{E}_k \left| \frac{W-1}{\delta_\infty} \right|^{2+\varepsilon} \leq \frac{C}{c_0-1} \mathbb{E}_k |W-1|^{2+\varepsilon} < \infty$  (see [15]). By the Berry-Esseen theorem, we have for all  $x \in \mathbb{R}$ ,

$$\left| \mathbb{P}_\xi \left( \frac{\Pi_n(W - W_n)}{\sqrt{Z_n} \delta_\infty(T^n \xi)} \leq x \right) - \Phi(x) \right| \leq C \mathbb{E}_{T^n \xi} \left| \frac{W-1}{\delta_\infty} \right|^{2+\varepsilon} \mathbb{E}_\xi [Z_n^{-\varepsilon/2}].$$

Taking expectation with  $Z_0 = k$  and using Theorem 2.1, we get

$$(4.1) \quad \begin{aligned} \left| \mathbb{P}_k \left( \frac{\Pi_n(W - W_n)}{\sqrt{Z_n} \delta_\infty(T^n \xi)} \leq x \right) - \Phi(x) \right| &\leq C \mathbb{E}_k \left| \frac{W-1}{\delta_\infty} \right|^{2+\varepsilon} \mathbb{E}_k [Z_n^{-\varepsilon/2}] \\ &\leq C A_{k,n}(-\varepsilon/2). \end{aligned}$$

□

**4.2. Large deviation rate for  $R_n$ .** This section is devoted to the proof of Theorems 2.9 and 2.10. Recall that  $M_{n,j}$  is defined by (2.42), where  $N_{n,i}$  are i.i.d. with generating  $f_n$ , given the environment  $\xi$  (see Section 2).

*Proof of Theorem 2.9.* Since for all  $n \in \mathbb{N}$ ,  $M_{n,j} - m_n$  has the same law as  $M_{0,j} - m_0$ , and is independent of  $Z_n$ , we obtain

$$\begin{aligned} \mathbb{P}_k(R_n - m_n \in D) &= \sum_{j \geq k} \mathbb{P}(M_{n,j} - m_n \in D) \mathbb{P}_k(Z_n = j) \\ &\leq \sum_{j \geq k} \frac{C_1}{j^r} \mathbb{P}_k(Z_n = j) \\ &= C_1 \mathbb{E}_k [Z_n^{-r}]. \end{aligned}$$

The result (2.43) follows from Theorem 2.1, and (2.45) follows similarly.  $\square$

*Proof of Theorem 2.10.* We start with a lemma which is a direct consequence of the Marcinkiewicz-Zygmund inequality (see [12, p. 356]).

**Lemma 4.1** ([21], Lemma 1.4). *Let  $(X_i)_{i \geq 1}$  be a sequence of i.i.d. centered r.v.'s. Then we have for  $p \in (1, \infty)$ ,*

$$(4.2) \quad \mathbb{E} \left| \sum_{i=1}^n X_i \right|^p \leq \begin{cases} (B_p)^p \mathbb{E}(|X_i|^p) n, & \text{if } 1 < p \leq 2, \\ (B_p)^p \mathbb{E}(|X_i|^p) n^{p/2}, & \text{if } p > 2, \end{cases}$$

where  $B_p = 2 \min \{k^{1/2} : k \in \mathbb{N}, k \geq p/2\}$  is a constant depending only on  $p$  (so that  $B_p = 2$  if  $1 < p \leq 2$ ).

We shall prove Theorem 2.10 in the case when  $p \in (1, 2]$ . Using the fact that  $M_{n,j} - m_n$  has the same law as  $M_{0,j} - m_0$  and is independent of  $Z_n$ , we obtain after conditioning

$$(4.3) \quad \mathbb{P}_k(|R_n - m_n| > a) = \sum_{j=k}^{\infty} \mathbb{P}(|M_{0,j} - m_0| > a) \mathbb{P}_k(Z_n = j).$$

Using (2.42) and the fact that, under  $\mathbb{P}_\xi$ , the r.v.'s  $N_{0,i} - m_0$  ( $i = 1, \dots, j$ ) are i.i.d. centered and with generating function  $f_0$ , we get from Lemma 4.1 that, for  $p \in (1, 2]$ ,

$$\begin{aligned} \mathbb{P}_\xi(|M_{0,j} - m_0| > a) &\leq a^{-p} \mathbb{E}_\xi |M_{0,j} - m_0|^p \\ &\leq \left( \frac{B_p}{a} \right)^p j^{1-p} \mathbb{E}_\xi |Z_1 - m_0|^p. \end{aligned}$$

Taking expectation, we obtain

$$\mathbb{P}(|M_{n,j} - m_n| > a) \leq \left( \frac{B_p}{a} \right)^p j^{1-p} \mathbb{E} |Z_1 - m_0|^p.$$



Therefore, coming back to (4.3) and applying Theorem 2.1, we get

$$\begin{aligned}
\mathbb{P}_k(|R_n - m_n| > a) &\leq \left(\frac{B_p}{a}\right)^p \mathbb{E}|Z_1 - m_0|^p \sum_{j=k}^{\infty} j^{1-p} \mathbb{P}_k(Z_n = j) \\
&= \left(\frac{B_p}{a}\right)^p \mathbb{E}|Z_1 - m_0|^p \mathbb{E}_k[Z_n^{1-p}] \\
&= C_p a^{-p} A_{k,n}(p-1),
\end{aligned}$$

with  $C_p = B_p \mathbb{E}|Z_1 - m_0|^p$ . This ends the proof of Theorem 2.10 in the case when  $p \in (1, 2]$ . The proof in the case  $p > 2$  is obtained in the same way.  $\square$

## REFERENCES

- [1] V. I. Afanasyev, C. Böinghoff, G. Kersting, and V. A. Vatutin. Limit theorems for weakly subcritical branching processes in random environment. *J. Theoret. Probab.*, 25(3):703–732, 2012.
- [2] V. I. Afanasyev, C. Böinghoff, G. Kersting, and V. A. Vatutin. Conditional limit theorems for intermediately subcritical branching processes in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(2):602–627, 2014.
- [3] K. B. Athreya and S. Karlin. Branching processes with random environments: II: Limit theorems. *Ann. Math. Stat.*, 42(6):1843–1858, 1971.
- [4] K. B. Athreya and S. Karlin. On branching processes with random environments: I: Extinction probabilities. *Ann. Math. Stat.*, 42(5):1499–1520, 1971.
- [5] K. B. Athreya and P. E. Ney. *Branching processes*, volume 28. Springer-Verlag Berlin, 1972.
- [6] V. Bansaye and J. Berestycki. Large deviations for branching processes in random environment. *Markov Process. Related Fields*, 15(4):493–524, 2009.
- [7] V. Bansaye and C. Böinghoff. Upper large deviations for branching processes in random environment with heavy tails. *Electron. J. Probab.*, 16(69):1900–1933, 2011.
- [8] V. Bansaye and C. Böinghoff. Lower large deviations for supercritical branching processes in random environment. *Proc. Steklov Inst. Math.*, 282(1):15–34, 2013.
- [9] V. Bansaye and C. Böinghoff. Small positive values for supercritical branching processes in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(3):770–805, 2014.
- [10] C. Böinghoff. Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions. *Stoch. Process. Appl.*, 124(11):3553–3577, 2014.
- [11] C. Böinghoff and G. Kersting. Upper large deviations of branching processes in a random environment - offspring distributions with geometrically bounded tails. *Stoch. Process. Appl.*, 120(10):2064–2077, 2010.
- [12] Y. S. Chow and H. Teicher. *Probability theory: independence, interchangeability, martingales*. Springer Science & Business Media, 2012.
- [13] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 2. Springer, 1998.
- [14] I. Grama, Q. Liu, and E. Miqueu. Asymptotic of the distribution and harmonic moments for a supercritical branching process in a random environment. *arXiv preprint arXiv:1606.04228*, 2016.
- [15] Y. Guivarc’h and Q. Liu. Propriétés asymptotiques des processus de branchement en environnement aléatoire. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, 332(4):339–344, 2001.
- [16] B. Hambly. On the limiting distribution of a supercritical branching process in a random environment. *J. Appl. Probab.*, 29(3):499–518, 1992.

- [17] C. Huang and Q. Liu. Moments, moderate and large deviations for a branching process in a random environment. *Stoch. Process. Appl.*, 122(2):522–545, 2012.
- [18] C. Huang and Q. Liu. Convergence in  $L_p$  and its exponential rate for a branching process in a random environment. *Electron. J. Probab.*, 19(104):1–22, 2014.
- [19] M. V. Kozlov. On large deviations of branching processes in a random environment: geometric distribution of descendants. *Discrete Math. Appl.*, 16(2):155–174, 2006.
- [20] M. V. Kozlov. On large deviations of strictly subcritical branching processes in a random environment with geometric distribution of progeny. *Theory Probab. Appl.*, 54(3):424–446, 2010.
- [21] Q. Liu. Local dimensions of the branching measure on a Galton–Watson tree. *Ann. Inst. Henri Poincaré Probab. Stat.*, 37(2):195–222, 2001.
- [22] M. Nakashima. Lower deviations of branching processes in random environment with geometrical offspring distributions. *Stoch. Process. Appl.*, 123(9):3560–3587, 2013.
- [23] P. E. Ney and A. N. Vidyashankar. Harmonic moments and large deviation rates for supercritical branching processes. *Ann. Appl. Probab.*, pages 475–489, 2003.
- [24] W. L. Smith and W. E. Wilkinson. On branching processes in random environments. *Ann. Math. Stat.*, 40(3):814–827, 1969.
- [25] D. Tanny. A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. *Stoch. Process. Appl.*, 28(1):123–139, 1988.
- [26] V. A. Vatutin. A refinement of limit theorems for the critical branching processes in random environment. In *Workshop on Branching Processes and their Applications. Lect. Notes Stat. Proc.*, volume 197, pages 3–19. Springer, Berlin, 2010.
- [27] V. A. Vatutin and X. Zheng. Subcritical branching processes in a random environment without the Cramer condition. *Stoch. Process. Appl.*, 122(7):2594–2609, 2012.

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